

RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

P.G. SECOND YEAR

M.A./M.SC. FOURTH SEMESTER (January – June), 2012

Mid-Semester Examination, March 2012

Date : 26/03/2012

Time : 11 am – 2 pm

MATHEMATICS

Full Marks : 75

[Use 4 Answer Books one for Gr-A (Unit-I), One for Gr-A(Unit-II), One for Gr-B and one for Gr-C]

Group – A

Unit - I

1. Attempt any two :

- Using Poisson Bracket, prove that the components of angular momentum of a particle moving in a central force potential are conserved. [4]
- Apply Hamilton-Jacobi Theory to one dimensional Hamiltonian $H = \frac{1}{2}(q^2 + p^2)$ to deduce the motion of the particle. [4]
- Using Poisson Bracket, show that the transformation given by $Q = \sqrt{e^{-2q} - p^2}$, $P = \cos^{-1}(pe^q)$ is canonical. Also find the generating function F_2 of the transformation. [4]

2. Attempt any two :

- Given the Canonical transformation $Q_j = Q_j(q, p, t)$ and $P_j = P_j(q, p, t)$ and the corresponding generating function $F_2 = F_2(q, P, t)$, show that $\left[Q_j, \frac{\partial F_2}{\partial t}\right] = \frac{\partial Q_j}{\partial t}$ Hence show that $\frac{dQ_j}{dt} = [Q_j, \bar{H}]$ [2½]
- If U and V are two dynamical variables, show that $\frac{d}{dt}[U, V] = \left[\frac{dU}{dt}, V\right] + \left[U, \frac{dV}{dt}\right]$ [2½]
- If the Hamiltonian of a system is $H = \frac{p^2}{2} - \frac{1}{2q^2}$, show that $F = \frac{pq}{2} - Ht$ is a constant of motion. [2½]

Unit - II

3. Answer any one question :

[6×1]

- Express $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite polynomials.
- Establish the following integral representation of Hypergeometric function

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt, \text{ if } \gamma > \beta > 0.$$

4. Answer any one question :

[6×1]

- Show that $u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)f(\varphi)d\varphi}{a^2 - 2ar \cos(\varphi - \theta) + r^2}$ is the solution of the Dirichlet problem $\nabla^2 u = 0$, $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$ with boundary condition $u(a, \theta) = f(\theta)$, $0 \leq \theta \leq 2\pi$.
- Find the finite and single valued solution for θ and φ of the partial differential equation

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} = 0 \text{ by the method of separation of variables.}$$

Group – B

[Answer as much as you can. The maximum you can score is 25]

Notations :

- ℓ_1 : Space of all summable sequences.
 - $\mathbb{R}^\infty$: Space of all real valued sequences.
 - $\mathbb{N}, \mathbb{Z}, \mathbb{R}, C[0,1]$ have usual meanings.
5. State whether true or false. Each question carries 2 marks except Q.No. (a). For full credit support your claim with appropriate reasoning or with the help of a counterexample :
- a) The metric space $(Q, d())$ is complete, where $d(q_1, q_2) = \begin{cases} 0 & \text{if } q_1 = q_2 \\ 1 + |q_1 - q_2| & \text{otherwise} \end{cases}$ [1]
- b) Any function $f : \mathbb{N} \rightarrow \mathbb{R}$ is continuous, when both the spaces are given usual metric.
- c) If $A = \{(x, y) : xy = 0\} \subset \mathbb{R}^2$, then $(A, \|\cdot\|_2)$ is complete.
- d) $(C[0,1], \|\cdot\|')$ is NLS where $\|f\|' = |f(0)|$.
- e) $(C[0,1], \|\cdot\|')$ is NLS where $\|f\|' = \sup_{x \in [0,1] \cap \mathbb{Q}} |f(x)|$.
- f) On the NLS $(c_{00}, \|\cdot\|_\infty)$ the linear operator $T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, \dots)$ is bounded.
- g) On the NLS $(c_{00}, \|\cdot\|_\infty)$ the linear operator $T(x_1, x_2, x_3, \dots) = \sum_n x_n$ is bounded.
- h) On the NLS $(C[0,1], \|\cdot\|_1)$ the linear operator $T(f) = f(0)$ is bounded.
- i) On the NLS $(C[0,1], \|\cdot\|_\infty)$ the linear operator $T_g(f)(x) = g(x)f(x)$ is bounded, where $g \in B[0,1]$.
- j) On the NLS ℓ^1 the rearrangement linear operator $T(x_1, x_2, x_3, \dots) = (x_2, x_1, x_4, x_3, \dots)$ is bounded.
- k) On the NLS ℓ^1 the linear operator $T(x_1, x_2, x_3, \dots) = \sum_n x_n$ is bounded.
- l) $((0,1], d')$ is complete where $d'(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$.
- m) The set $\mathbb{Q} \times \mathbb{N}$ is nowhere dense in $\mathbb{R}^2$ with usual Euclidean norm.
6. On $\mathbb{R}^\infty$ consider the following metric given by $d(x, y) = \frac{1}{n}$, where $x_i = y_i$; $1 \leq i < n$, and $x_n \neq y_n$. So this is the first time a mismatch occurs. Check that this is a valid metric. Find the closure of c_{00} with respect to this metric. [2+3]

Group – C

[Answer Part – I or Part – II]

Part – I (Operation Research)

7. Answer any one question : [9×1]
- a) Prove that the Karush-Kuhn-Tucker condition are both necessary and sufficient for optimality of a linear programming problem.
- b) Consider the inventory models involves constant rate demand with instantaneous order replenishment and shortage is allowed
- Let y = Order quantity (number of units)
 D = demand rate (units per unit time)
 w = Maximum shortage
 k = setup cost associated with the placement of an order
 h = Holding cost per inventory unit per unit time

p = Penalty cost per unit per unit time

Show that—

$$TC(y, w) = \frac{kD}{y} + \frac{h(y - w)^2 + pw^2}{2y}$$

$$y^* = \sqrt{\frac{2kD(p + h)}{ph}}$$

$$w^* = \sqrt{\frac{2kDh}{p(p + h)}}$$

8. Answer any two questions :

[6×2]

a) Consider the following problem

Maximize $2x_1 + 3x_2 + 5x_3$

Subject to $x_1 + x_2 + 2x_3 \leq 8$

$x_1 - x_2 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

i) Find the optimal solution

ii) Find the new optimal solution if the cost coefficient C_2 changes from 3 to -4.

b) The following simplex tableau shows the optimal solution of a linear programming problem. It is known that x_4 and x_5 are the slack variables in the first and second constraints of the original problem. The constraints are of the $\leq$ type

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-2	0	-3	-2	-35
x_3	0	0	$\frac{1}{4}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_1	0	1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{2}$

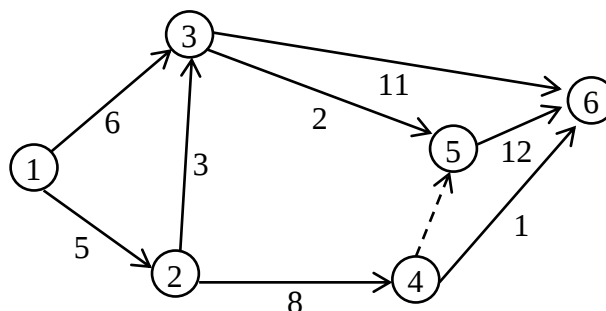
i) Write the original problem

ii) What is the dual of the original problem?

c) Find the sequence that minimizes the total time required for performing the following jobs on three machines in the order ABC.

Jobs	Machine A	Machine B	Machine C
1	8	3	8
2	3	4	7
3	7	5	6
4	2	2	9
5	5	1	10
6	1	6	9

d) Determine the critical path for the project network in Figure given below. All the durations are in days.



Also calculate : Total Float, Free Float and Independent Float.

9. Answer any one question :

[4×1]

a) Consider the linear programming problem

$$\text{Maximize } x_1 + x_2$$

$$\text{Subject to } 2x_1 - 3x_2 \leq 6$$

$$-3x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Discuss the feasibility of the dual problem.

b) Consider the following problem

$$\text{Maximize } 10x_1 + 24x_2 + 20x_3 + 20x_4 + 25x_5$$

$$\text{Subject to } x_1 + x_2 + 2x_3 + 3x_4 + 5x_5 \leq 19$$

$$2x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 57$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

i) Write the dual problem and verify that $(y_1, y_2) = (4, 5)$ is a feasible solution

ii) Use the information in part (i) to derive an optimal solution to both the primal and the dual problem.

c) Two inventory policies have been suggested by the purchasing department of a company.

Policy 1 : Order 150 units. The reorder point is 50 units and the time between placing and receiving an order is 10 days.

Policy 2 : Order 200 units. The reorder point is 75 units and the time between placing and receiving order is 15 days.

i) Which of the two policies should the company adopt?

ii) If you were in charge of devising an inventory policy for the company, what would you recommend assuming that supplier required a lead time of 22 days.

Given : The set up cost per order = Rs. 20.

The holding cost per unit inventory per day is Re 0.02.

Part – II (Algebraic Topology)

10. Answer any five questions :

[5×5]

a) i) Prove that for each $f \in C(X)$, there is $g \in C^*(X)$ such that $Z(f) = Z(g)$.

ii) Prove that $Z(X)$, the family of all zero-sets in X is closed with respect to countable intersection. [2+3]

b) i) Define completely separated sets in a space X . Prove that two sets in X are completely separated if and only if they are contained in disjoint zero-sets.

ii) Is $\mathbb{N}$ C^* -embedded in $\mathbb{R}$? Justify. [(1+3)+1]

c) In $C(X)$, show that all positive units have the same number of square roots.

d) Prove that X is connected if and only if 1 has exactly two square roots.

e) Define a pseudo-compact space. Construct an example of a pseudo-compact space which is not compact. [1+4]

f) In $C(X)$, prove that every prime ideal is contained in a unique maximal ideal.

g) i) Define a z -ideal in $C(X)$. Show that every z -ideal in $C(X)$ is an intersection of prime ideals.

ii) In $C(\mathbb{N})$, prove that every ideal is a z -ideal. [(1+2)+2]

h) If X is a finite discrete space, prove that every ideal in $C(X)$ is generated by an idempotent.

